Simulation of Long-term Returns with Stochastic Correlations

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ABSTRACT

This paper focuses on a nonlinear stochastic model for financial simulation and forecasting based on assumptions of multivariate stochastic correlation, with an application to the European market. We present in particular the key elements of a structured hierarchical econometric model that can be used to forecast financial and commodity markets relying on statistical and simulation methods. The investment universe includes money-market, fixed-income, inflation-linked bonds as well as equity and commodity indices. For each such investment opportunity a dedicated statistical model has been developed to generate future return paths describing the uncertainty the investment manager is facing over time.

Keywords:

Multivariate statistical method, Stochastic correlation, Monte Carlo simulation.

1. Introduction

A crucial issue for consistent financial management is an accurate econometric modeling of market conditions, sometimes over long-term horizons. The dynamic behavior of market values is dependent on random factors such as interest and inflation rates, the economic cycle and correlation among assets.

The recent sovereign crisis in Europe (2009-2011) has deeply affected previous financial markets conditions leading to a phase of unprecedented monetary easing and low interest rates, which unlike in the past, were not effective in recovering market stability and economic growth. In particular the European market suffered from increasing default risk of selected sovereign borrowers and increasing correlation and systemic risks in the Euro zone. In this situation, investments, which had long-term horizon plans, were faced with a downside potential of the financial markets in many countries. During such periods of market fluctuation, the adoption of a dynamic asset return model for securities exposed to market risk may play a crucial role. By focusing on market prices, we propose in this work a simulation approach to incorporate stochastic correlations in a two-layer multivariate statistical model.

We consider a simulation approach based on a hierarchical econometric model with EU GDP and inflation rate considered as the basic driving forces from which further economic factors, such as short and long-term interest rate and asset returns, are derived. The financial risk factors and asset returns are modeled with parameters and correlations fitted to quarterly data.

In this article, we develop and test a long-term asset returns simulation method in a two-level fashion with a dynamic stochastic correlation model and relying on a realistic asset universe, with interest-rate sensitive assets, such as government and inflation linked bonds, money market fund, equity, commodities. The stochastic model presented in section 1 extends [6], where a stochastic mean and constant correlation model was employed in an individual asset-liability management problem. Several statistical models are currently in use to simulate asset price and returns for the solution of decision problems under uncertainty. We refer to [21] which was among the first to introduce a cascade structure for asset price return simulation, later adopted by [11] with CASM (Cascade Asset Simulation Model) and [1]. A practical approach to include an assumption of stochastic correlations in a two-layer asset simulation and test its effectiveness are the main objectives of this article.

Stochastic correlation was introduced as dynamic conditional correlation (DCC) model by [16] with analyzing the performance of the model for large covariance matrices. DCC method has been used in the econometric and financial problems. We cite some of the most relevant contributions [12, 3, 13, 22, 19, 23, 15, 9].

The article is structured in 4 sections. In section 2 we introduce stochastic correlation and long-term asset returns simulation in a two-layer structure. In section 3 we apply the model and show the returns simulation results and in Section 4 we conclude

2. Asset Returns Simulation

With simulation techniques, many possible future situations of a financial portfolio can be evaluated. In financial planning the unfolding uncertain future is represented by a large number of future paths from the simulation process which uses Monte Carlo simulation technique [17] to investigate the evolution of asset returns over planning horizon. This is an advantage of the simulation that we can use a relatively large number of scenarios for the future circumstances.

For any financial planning problem, we need to know about behavior of our assets over time horizon. Since market conditions are changing over time and exposed to stochastic factors, we introduce an asset returns model in discrete-time with a modeling framework based on Two-Layers Asset Simulation Model (TLASM) with stochastic correlation among assets. The multivariate statistical model can reflect market dynamics during instability periods.

2.1. Stochastic correlations

As stated in [12], correlations are critical inputs for common tasks of financial management. A forecast of future correlations and volatilities is the basis of any pricing formula. Asset allocation and risk assessment also rely on correlations, however in this case a large number of correlations are often required. Construction of an optimal portfolio with a set of constraints requires a forecast of the covariance matrix of the returns. Similarly, the calculation of the standard deviation of today's portfolio requires a covariance matrix of all the assets in the portfolio. Simple methods such as rolling historical correlations and exponential smoothing are widely used. More complex methods such as varieties of multivariate GARCH or Stochastic Volatility have been extensively investigated in the econometric literature [20, 16, 2, 13].

We consider a *dynamic conditional correlation* (DCC) model [12] and then present an approach for asset returns simulation consistent with such assumption. Let $r_{i,t}$ be the return of asset $i \in I$ at time $t \in T$. We indicate with r_t the return vector with components $r_{i,t}$. Given r_0 and $\omega \in (\Omega, F, P)$ as a generic random source of risk, for t=1,2,...,T, we define the stochastic differential equations for return process as:

$$r_t = r_{t-dt} + \mu(t,\omega)dt + \Gamma(t,\omega)\sqrt{dt}e(\omega) \quad (1)$$

where $\mu(t,\omega)dt$ is an instantaneous drift. We assume in section 1.2 dedicated models for different asset indexes. $\Gamma(t,\omega)$ is a random covariance matrix that admits decomposition $\Gamma_t = D_t C_t D_t$ and $e \sim N(0,1)$. D_t is a diagonal matrix with elements $\sigma_{i,t}$ with $i = 1,2,...,I$, includes the I returns' conditional standard deviations which can be defined by any type of a univariate GARCH process, while $C_t := \{c_{ij,t}\}$**1.** includes the time-varying correlation coefficients between asset i and asset j at time t.

The DCC is a natural extension of the GARCH models. In the DCC model [12] the relationship between conditional correlations and conditional variance was obtained expressing the returns, $r_{i,t}$ as

$$r_{i,t} = \sigma_{i,t}\varepsilon_{i,t} \quad \textit{where} \quad \varepsilon_{i,t} \sim N(0,1). \quad (2)$$

We assume a *Threshold Autoregressive Conditional Heteroskedasticity* (TARCH) process of the first order for $\sigma_{i,t}^2$. [18] and [24] introduced independently the TARCH models which allows for asymmetric shocks to volatility. According to [14], TARCH volatility process is a way of parametrizing the sign of the innovation that may influence the volatility in addition to its magnitude.

In DCC model [12], the conditional correlation matrix is modeled as:

$$C_t = \tilde{Q}_t^{-0.5} Q_t \tilde{Q}_t^{-0.5} \quad (3)$$

Where $Q_t = \{q_{ij,t}\}$ is the conditional covariance matrix and $\tilde{Q}_t$ is the diagonal matrix of the *i* th diagonal element of the Q_t. Dynamics of Q_t can be consider with following the DCC model [12] assumption as:

$$Q_t = (1-\alpha-\beta)\bar{Q} + \alpha\varepsilon_{t-1}\varepsilon'_{t-1} + \beta Q_{t-1} \quad (4)$$

where $\bar{Q}$ is the unconditional correlation matrix of ε. α and β are constant parameters such that $\alpha, \beta \geq 0$ and $\alpha + \beta < 1$ to ensure positive definiteness and stationarity, respectively. Accordingly, the variance-covariance process dynamics can be expressed as

$$\Gamma_t = D_t(\tilde{Q}_t^{-0.5} Q_t \tilde{Q}_t^{-0.5})D_t \quad (5)$$

2.2. Two-Layers Asset Simulation Model

According to the figure 1, in this model we consider two layers structure, at the first layer we derive the relative risk factors then in the second layer compute the equity risk premium and returns of each asset class.

Risk factors have essentially affect in the long-term financial position of the portfolio return and need to be consider in the economic and financial modeling framework. Therefore, for the given investment universe we need to identify associated risk factors relevant to the set of investment opportunities.

Structure of the risk model includes all market variables adopted to derive the benchmarks' evolution: money market benchmark, bond indexes, equity and commodity benchmarks [21, 4, 8, 1, 11, 10, 6].

Table 1 includes Euro area market benchmark indexes and the associated risk factors of each investment opportunities in this study. We consider inflation rate, GDP output gap and 12-month Euribor together with 10 years Euro interest rate as short and long-term interest rate in our risk factor modeling. Inflation and interest rates are two critically important risk-factors for any financial investment. In particular assets can suffer of inflation shocks because an inflation increase normally triggers an interest rate upshift thus negatively hitting the fixed income nominal assets. We refer here to the risk process as the random process of the financial factors embodying the risk sources of the problem [8, 5, 6].

We translate a set of continuous stochastic dynamics into a discrete event structure relies on a scenario tree labeling

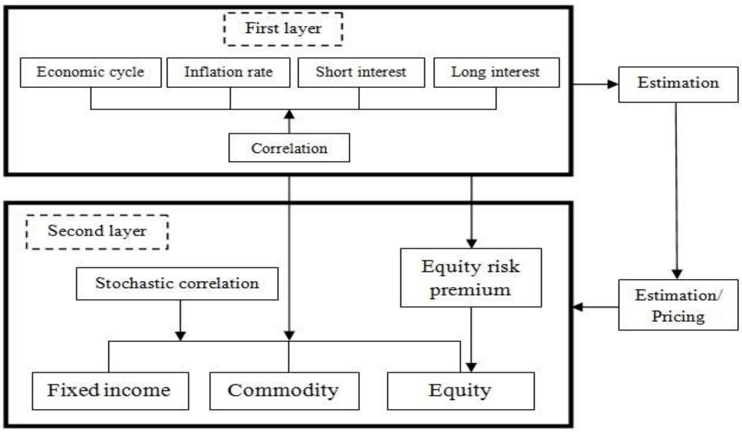


Figure 1: Structure of two layers asset simulation model

scheme [7]. Nodes along the tree, for $t \in T$, are denoted by $n \in N_t$. For $t = 0$ the root node (associated with the partition $N_0 = \{\Omega, \varnothing\}$, corresponding to the entire probability space) is labeled $n = 1$.

For $t > 0$ every $n \in N_t$ has a unique ancestor n^- and, for $t < T$, a non-empty set of children nodes n^+. We denote with t_n the time associated with node n. The set of all predecessors of node n : $n^-, n^{--}, \ldots, 0$ is denoted by P_n. From the statistical modeling viewpoint, first layer risk factors $\zeta_{j,n}$ plus equity risk premium λ_n and then second layer asset price return $r_{i,n}$ are computed with multivariate Gaussian return model with autoregression and exogenous variables. Stochastic difference equation for risk factors $\zeta_{j,n}$ for $j = 1,2,3,4$ and equity risk premium λ_n for all $n \in N_t$ are considered as follow:

$$\zeta_{j,n} = \mu_{j,n} + \sigma_j \sqrt{\Delta t} \sum_{r \geq j} c_{j,r} e_n^r$$

(6)

where vector $\mu_{j,n}$ gives the following stochastic drift of the risk factors at nod n :

$$\mu_{j,n} = \begin{cases} \beta_0^1 + \beta_1^1 \zeta_{1,n^-} + \beta_2^1 \zeta_{1,n^{--}} & j = 1 \\ \beta_0^2 + \beta_1^2 \zeta_{2,n^-} + \beta_2^2 \zeta_{1,n} + \beta_3^2 \zeta_{1,n^-} + \beta_4^2 \zeta_{1,n^{--}} & j = 2 \\ \beta_0^3 + \beta_1^3 \zeta_{2,n} + \beta_2^3 \zeta_{1,n} & j = 3 \\ \beta_0^4 + \beta_1^4 \zeta_{4,n} + \beta_2^4 \zeta_{1,n} + \beta_3^4 \zeta_{2,n} & j = 4 \end{cases}$$

(7)

$$\lambda_n = \beta_0^\lambda + \beta_1^\lambda \zeta_{4,n^-} + \beta_2^\lambda \zeta_{2,n^-} + \sigma^\lambda \sqrt{\Delta t} e_n$$

(8)

Table 1: Market benchmarks and relative risk factors

Investment Class	Benchmark	Risk factors
Money market	Euribor 12 month	Inflation rate, economic cycle, 12-month short rate
European Gov. bond	JPM Global Emu	Euro stock, interest rate, economic cycle
-	JPM Global ex-EMU	Euro stock, inflation rate
Inf. Link. bond	Barclays Infl. Linked	Euro stock, interest rate, inflation rate
Equity	MSCI Europe Index	Stock risk premium, interest rate
Commodity	Dow Jones Comm.	Euro stock, economic cycle, euro bond market

The risk factors ζ_j for *j* = 1,2,3,4 defined as: ζ_1 is the GDP output gap, ζ_2 is the inflation rate, ζ_3 is the short interest rate and ζ_4 is the long interest rate. In the model, Δt defines the time increment between nodes n^- and *n*. Risk factors correlation is introduced directly on the realizations e_n^r of four standard normal variables through the Cholesky elements $c_{j,r}$ of the correlation matrix with normal distribution $N(0,1)$ and illustrated in Table 3.

The set of estimated coefficients and risk factors provides an input to the generate the asset returns scenarios and will determine the returns' evolution over the long-term decision horizon. The asset returns simulation of each benchmark $r_{i,n}$ in each node $n \in N_t$ for scenario generation can now be determined as following formulas for the different benchmark indexes:

Europe bonds:

$$r_{i,n} = \beta_0^i r_{i,n^-} + \mu_{i,n} + \sigma_{i,n}\sqrt{\Delta t}\sum_r c_{i,r,n} e_n^r$$
$$i = 2,3,4$$
(9)

where vector $\mu_{i,n}$ defined stochastic drift at node *n* as follows:

$$\mu_{i,n} = \begin{cases} \beta_1^2 + \beta_2^2 r_{5,n} + \beta_3^2 \zeta_{1,n} + \beta_4^2 \zeta_{4,n} & i = 2 \\ \beta_1^3 + \beta_2^3 r_{5,n} + \beta_3^3 \zeta_{2,n} & i = 3 \\ \beta_1^4 + \beta_2^4 \zeta_{2,n} + \beta_3^4 r_{5,n^-} + \beta_4^4 \zeta_{4,n} & i = 4 \end{cases}$$
(10)

For the European bonds r_i (*i* = 2,3,4), we consider r_2 as JPM Global Government Bond EMU index, r_3 for the JPM Global Government Bond ExEMU index and ζ_4 for the Barclays Euro Inflation Linked bond index.

In the statistical models, coefficients show the dependence of the government bonds return with the stock risk premium.

Equity:

• MSCI Europe equity

$$r_{5,n} = (r_{1,n^-} + \lambda_n) + \sigma_{5,n}\sqrt{\Delta t}\sum_r c_{i,r,n} e_n$$
(11)

The equity return is modeled relying on the performance of the equity risk premium (see equation 8) and short-term interest rate in Euro zone. The risk premium is depended on the long-term interest rate, inflation rate and economic cycle.

Indirect real asset:

• Commodities

$$r_{6,n} = \beta_1^6 + \beta_2^6 r_{5,n} + \beta_3^6 r_{4,n^-} + \beta_4^6 \zeta_{1,n^-} + \beta_5^6 \zeta_{2,n} + \sigma_{6,n}\sqrt{\Delta t}\sum_r c_{6,r,n} e_n^r$$
(12)

The commodity return is evaluated by a stable relationship with the GDP output gap, inflation rate, equity market and Euro inflation linked bond yields.

2.3. Statistical Assumptions and Estimation

The economic and financial risk factors are modeled with parameters and correlations fitted to quarterly data in a two level TLASM fashion.

We use delegate total return indices for the asset classes. A total return index includes the reinvestment of dividends in the case of stock markets and the gains or losses of the price variation in the case of bond markets, respectively. All of the mentioned data sets have been collected at a quarterly frequency through the Data-Stream source. Furthermore, for all data sets, the time-specified of past observed market data starts from January 1999 and ends in December 2015. The statistical coefficients are estimated through the method of ordinary least squares (OLS) (see table 2 and 3). Risk factors correlation matrix estimated and reported in Table 4.

3. Simulation Results

The numerical results have been computed through a MATLAB R2014a for simulation and Excel as the input and output data collection. The simulation has been generated with 2048 number of paths at the horizon.

We consider a discrete simulation model over the 10-year planning horizon from Q1 2009 include investment universe based on the table 1.

To show the complex nature of the output from TLASM with stochastic correlation, forward scenario tree generation of the corresponding asset class returns have been performed by back-tested analysis with quarterly data from Q1 2009 to Q4 2016.

We show in figure 2 a set of output trees generated for representative asset index classes based on the introduced simulation model with stochastic -right side- versus constant correlation -left side- models. The simulation is generated across time with respect to observed market dynamic at each stage up to Q4 2015. The different techniques for simulation are ex-post analyzed on actual market dynamic with same estimated coefficients and assumption.

We consider the periods of 2009 until 2016 which crisis, post crisis and recent market situation took place. In almost all cases the asset returns simulation with stochastic correlation includes actual market value over the simulation period seen to that date which includes also recent market instability.

Table 2: Risk factors estimated coefficients and volatilities

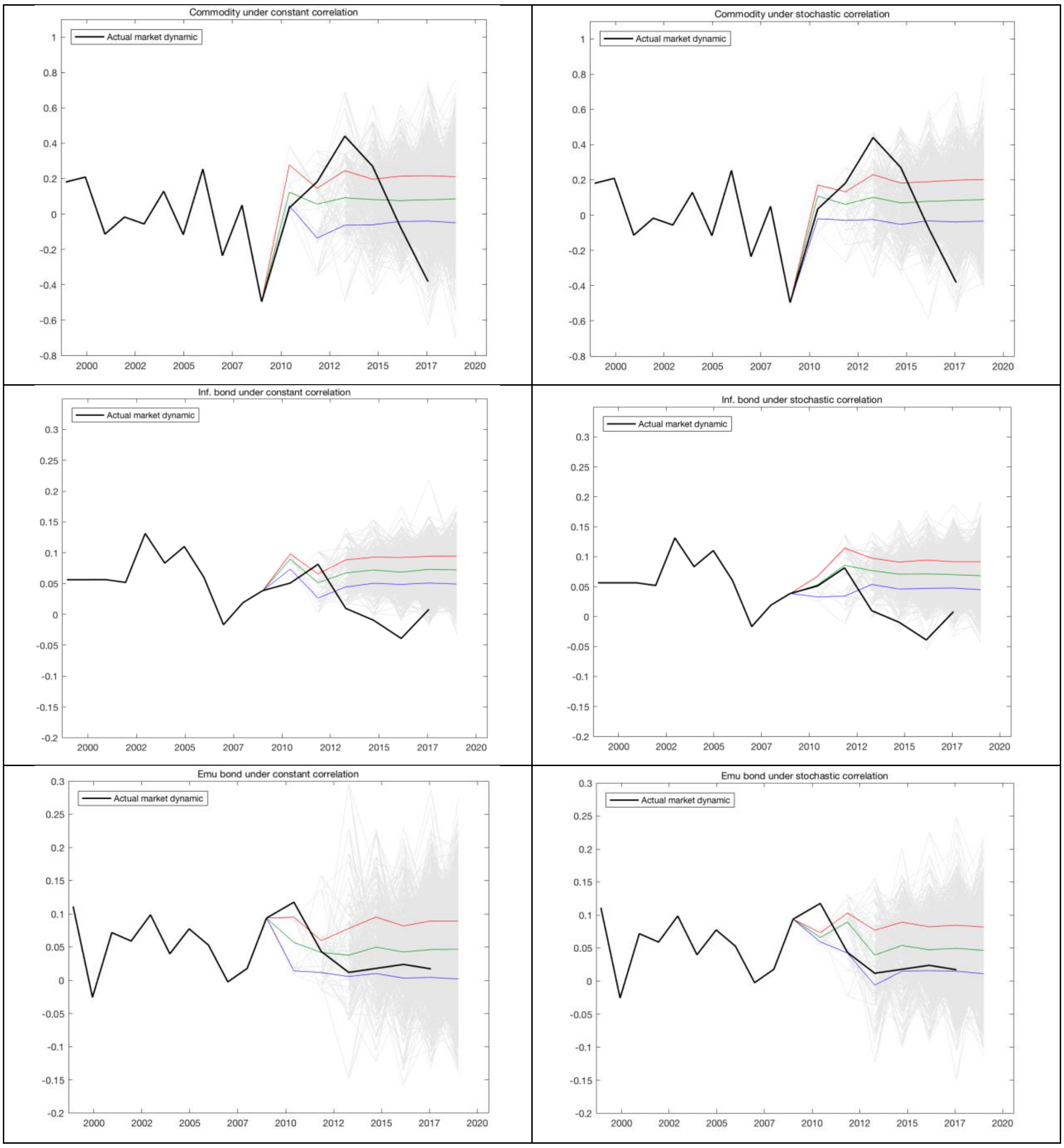
	β_0	β_1	β_2	β_3	β_4	Vol. (%)
Out-put gap	0.0002	1.8095	-0.8950	–	–	1.14
Inflation	0.0123	0.4317	-0.6236	1.4184	-0.8012	0.92
Short rate	0.0391	-0.2483	0.5889	–	–	1.53
Long rate	0.0191	0.7749	0.0873	-0.4572	–	1.41
Risk premium	1.0564	-13.2855	-21.2597	–	–	20.07

Table 3: Risk factors correlation matrix

	Long rate	Inflation	Out-put gap	Short rate
Long rate	1.0000	0.6775	0.4203	0.8338
Inflation	0.6775	1.0000	0.2336	0.6075
Out-put gap	0.4203	0.2336	1.0000	0.6379
Short rate	0.8338	0.6075	0.6379	1.0000

Table 4: Estimated coefficients for asset return models

	β_0	β_1	β_2	β_3	β_4	β_5
Bond EMU	0.3470	0.1126	-0.1293	-0.7485	-1.6853	–
Bond ex-EMU	0.4212	0.0483	-0.0659	-0.8926	–	–
Infl. Link. bond	–	0.1635	-0.2051	-0.1344	-1.9173	–
Commodity	–	-0.0018	0.5862	1.3496	4.7175	-3.039



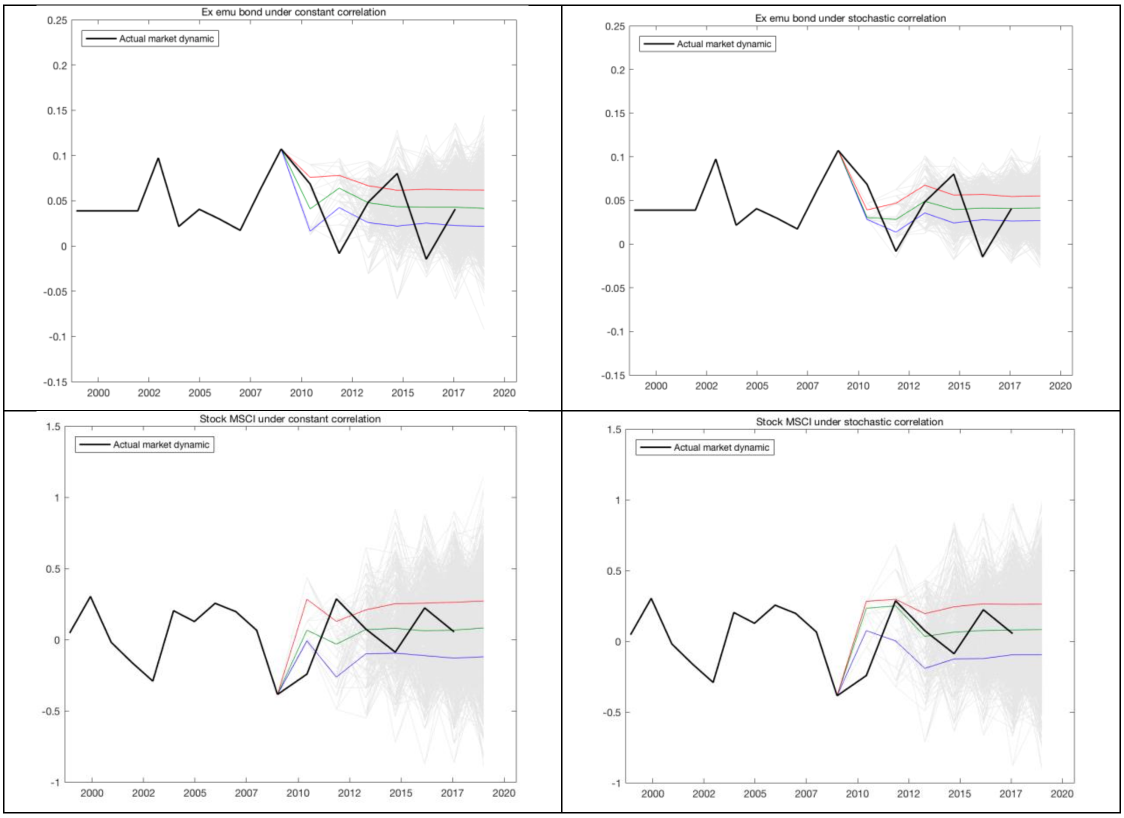


Figure 2: Asset returns scenario simulation of investment universe under constant correlation -left side and stochastic correlation -right side- (green trajectory is the mean scenario, purple and red trajectories 25% and 75% quantile of the distribution, respectively) with components related to the modelling of time series for commodities and financial markets.

4. Conclusion

We have presented a simulation approach for long-term returns that, building on concepts from existing methods, integrates a set of economic and statistical requirements in a computationally efficient scheme. The method has been developed based on the idea of two-layer simulation model using stochastic correlations for financial problems, which may lead to a sufficient representation of the randomness underlying the decision-making process. This technique has been tested on actual market dynamic and we report evidence of its effectiveness compare to the method with constant correlation. Since there is correlation clustering changing dynamically during and after the crisis therefore introducing stochastic correlation among assets into simulation method is needed in order to approximate return distribution. We presented effectiveness result of returns distribution under stochastic correlation versus constant correlation. The numerical evidence supports this method to yield a good approximation of a distribution with a long-term planning horizon. The methodology can be adopted for the formulation of optimization problems in financial and commodities markets.

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