



## Jump Diffusion & Stochastic Volatility Models for Option Pricing (Application in Python & MATLAB)

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### ABSTRACT

The Black-Scholes model assumes that the price of the underlying asset follows a geometric Brownian motion. This assumption has two implications: first, log-returns over any horizon are normally distributed with constant volatility  $\sigma$  and the second, stock price evolution is continuous, therefore, there is no market gaps. These conditions are commonly violated in practice: empirical returns typically exhibit fatter tails than a normal distribution, volatility is not constant over time, and markets do sometimes gap. The existence of volatility skew will misprice options price. Derived from these flaws, a number of models have proposed. In this paper we will analyze, simulate and compare two most important models which have widespread using: jump diffusion model and stochastic volatility model. Each of the aforementioned models have programmed in MATLAB and Python, then their results have been compared together in order to provide a robust understanding of each of them. Our results show that in comparison to Black-Scholes model these two models yield better performance.

### Keywords

Jump-Diffusion Model, Stochastic Volatility Model, Black-Scholes Model.



## 1. Introduction

The Black-Scholes is one of the crucial models in financial engineering area. It was developed in 1973 by Fischer Black, Robert Merton, and Myron Scholes and is still widely used. It is considered as one of the best ways to specify options prices. The model requires five input variables: time to expiration, the current stock price, the risk-free rate, the volatility and the strike price of an option (Golbabai et al. 2019; Ma et al. 2020).

The model assumes stock prices follow a lognormal distribution because asset prices should be positive. Often, asset prices are observed to have significant right skewness and some degree of kurtosis (fat tails). This means high-risk downward moves often happen more often in the market than a normal distribution predicts.

The assumption of lognormal underlying asset prices should thus show that implied volatilities are similar for each strike price according to the Black-Scholes model. However, since the market crash of 1987, implied volatilities for at the money options have been lower than those further out of the money or far in the money. The reason for this phenomenon is the market is pricing in a greater likelihood of a high volatility move to the downside in the markets (Tian and Zhang 2020).

This has led to the presence of the volatility skew. When the implied volatilities for options with the same expiration date are mapped out on a graph, a smile or skew shape can be seen. Thus, the Black-Scholes model is not efficient for calculating implied volatility.

The Black-Scholes model makes certain assumptions:

- The option is European and can only be exercised at expiration.
- No dividends are paid out during the life of the option.
- Markets are efficient (i.e., market movements cannot be predicted).
- There are no transaction costs in buying the option.
- The risk-free rate and volatility of the underlying are known and constant.
- The returns on the underlying are normally distributed (Li et al. 2019).

As it is clear, Black-Scholes model makes some restrictive assumptions, which are not necessarily exist in the real world. The variety of models have been offered to modify the Black-Scholes model, some of them are mentioned here:

(a) chaos theory fractal Brownian motion, and stable processes; for example, Mandelbrot (1963), Rogers (1997), Samorodnitsky and Taqqu (1994); (b) generalized hyperbolic models, including log t model and log hyperbolic model; for example, Barndorff-Nielsen and Shephard (2001), Blattberg and Gonedes (1974); (c) time-changed Brownian motions; for example, Clark (1973), Madan and Seneta (1990), Madan et al. (1998), and Heyde (2000). An immediate problem with these models is that it may be difficult to obtain analytical solutions for option prices.

In a parallel development, different models are also proposed to incorporate the “volatility smile” in option pricing. Popular ones include: (a) stochastic volatility and ARCH models; for example, Hull and White (1987), Engle (1995), Fouque et al. (2000); (b) constant elasticity model (CEV) model; for example, Cox and Ross (1976), and Davydov and Linetsky (2001); (c) normal jump models proposed by Merton (1976);

(d) affine stochastic-volatility and affine jump-diffusion models; for example, Heston (1993), and Duffie et al. (2000); (e) models based on Lévy processes; for example, Geman et al. (2001) and references therein; (f) a numerical procedure called “implied binomial trees”; for example, Derman and Kani (1994) and Dupire (1994).

The following table provides detailed explanation for prominent models which offered to modify Black-Scholes model.

Jump diffusion and stochastic volatility models answer to two major flaws in Black-Scholes model. Jump diffusion considers discontinuities in observed price process and stochastic volatility regards volatility as a mutable variable unlike Black-Scholes. The main contribution of this paper is its goal to explain in detail those two models and simulate them with different scenarios using Python and MATLAB to pave the way for scholars to make a deep comprehension about them. Furthermore, we compare performance of Black-Scholes and aforementioned models for option pricing.

**Table 1: Models proposed to modify flaws of Black-Scholes model**

Jump-Diffusion Model	Merton for the first time generalized Black-Scholes Model (1976). Merton assumes that stock returns are composed of two parts: Normal Part & Extraordinary Part Normal part progress according to the Geometric Brownian Motion process Extraordinary Part: causes unexpected jumps in stock price. The probability of jumps and the distribution of the jump size will enter in the option pricing problem. Jumps are market gaps and result in fat tails in return distribution.
Stochastic Volatility Models	These models are second generalization of Black-Scholes Model. Heston published crucial and influential paper in this area (1993). Although in Black-Scholes Model the volatility is constant, in this model we consider it as a variable. Volatility is a random variable that progresses over time. The correlation between changes in volatility and returns enters as another important variable. Stock prices are continuous in these models. Stochastic volatility results in fat tails.
AERCH/GARCH Models	Volatility may not be separately random but may change over time in a manner that may be dependent on the movement of stock price. Engle (1982) developed ARCH. Bollerslev (1986) proposed GARCH. Duan (1995) developed option pricing theory for these models
Non-Normal Models	Among the other approaches that have been proposed to better fit observed option prices are those that directly posit non-normal returns distribution. These include the log stable models of Carr and Wu (2003) and the variance gamma model of Madan, Carr and Chang (1998).

## 2. Jump-diffusion model

A market gap is a discontinuous price move. The Black-Scholes Model does not consider these discontinuities but empirical evidence shows markets do gap, especially when unexpected good or bad news hit the market. Robert Merton (1976) modified the Black-Scholes Model by adding a jump process to it. Jump process is a process that remains constant between jumps and changes at jump times. The stock price process in Merton's framework consists of two processes. One based on a GBM (Geometric Brownian Motion) process and another on a jump process. To specify jump diffusion model we should determine 1. The GBM process including its volatility 2. Frequency of jumps 3. The distribution of jump size. For calculating frequency of jumps we use Poisson distribution. The Poisson distribution is frequently used to represent random arrivals. The probabilities of the outcomes are defined by:

$$\text{Probability (N = K)} = \frac{e^{-\lambda} * \lambda^k}{k!}$$

The mean and variance of Poisson distribution are:

$$E(N) = \sum_{K=0}^{\infty} [K * \text{Probability (N = K)}] = \lambda$$

$$\text{Var}(N) = E(N^2) - [E(N)]^2 = \lambda$$

### 2.1. Jump diffusion returns specification

Here we use Poisson distribution in order to calculate number of jumps in the stock price. One of the crucial assumption of Black-Scholes Model is that log- returns over (0, t) equals to normally-distributed random variable. This means,

$$R_t = \ln\left(\frac{S_T}{S_0}\right)$$

$$R_t = Z_t$$

However, in a jump diffusion model we have additional part in the second equation and that is the outcomes of each of a random number of jumps. We should calculate the number of jumps by Poisson distribution. But first of all we should modify Poisson

distribution in order to scale the distribution with the length of the horizon since jump should become more likely over a longer horizon. Number of jumps in the interval (0, t) is denoted by  $N_t$ .

$$\text{probability } (N_t = K) = \frac{e^{-\lambda t} * (\lambda t)^k}{k!}$$

Furthermore, we should specify how the jump returns are distributed. Merton (1976) assumed that each jump return is normally distributed and jump outcomes are independent of each other. So we denote a sequence of independent and identically distributed random variables by  $(H_k)$ . conditional on there being K jumps in the interval (0, t), the returns  $R_t$  are given by:

$$R_t = \begin{cases} Z_t & \text{if } K = 0 \\ Z_t + H_1 + H_2 + \dots + H_k & \text{if } K \geq 1 \end{cases}$$

Das and Sundaram (1999) show that the first four moments of  $R_t$  are:

$$\text{Mean} = (\alpha + \lambda \mu) t$$

$$\text{Variance} = [\sigma^2 + \lambda(\mu^2 + \gamma^2)] t$$

$$\text{Skewness} = \frac{1}{\sqrt{t}} \left[ \frac{\lambda(\mu^3 + 3\mu\gamma^2)}{((\sigma^2 + \lambda(\mu^2 + \gamma^2))^{\frac{3}{2}})} \right]$$

$$\text{Kurtosis} = 3 + \frac{1}{t} \left[ \frac{\lambda(\mu^4 + 6\mu^2\gamma^2 + 3\gamma^4)}{((\sigma^2 + \lambda(\mu^2 + \gamma^2))^2)} \right]$$



Figure 1: Simulated distributions of returns from jump diffusion model

The following figures show sample of distributions of returns from jump diffusion model which are selected from simulated distributions of returns in Python with

corresponding parameters and specified features above:

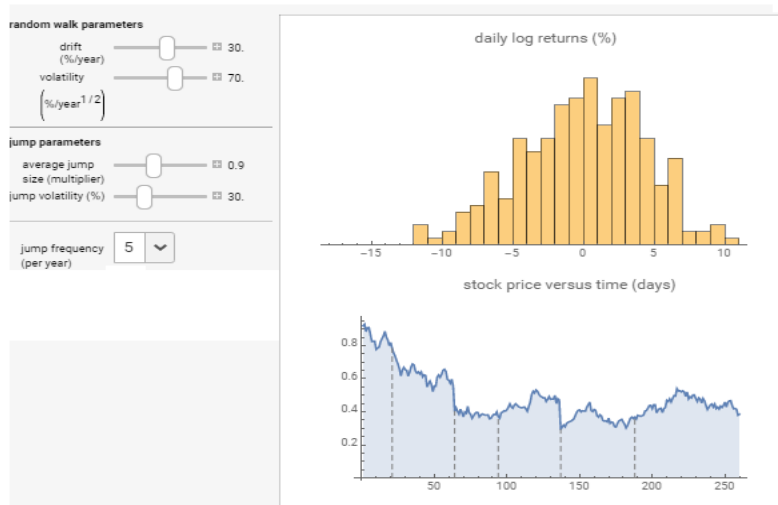


Figure 2: Simulated distributions of returns from jump diffusion model

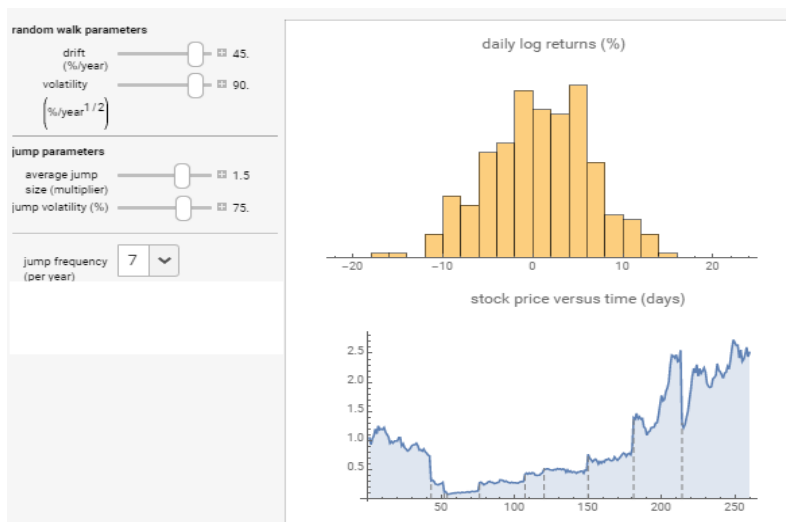


Figure 3: Simulated distributions of returns from jump diffusion model

### 2.2. Merton option pricing

Let  $S$  be the current stock price and  $r$  be the risk free rate of interest. By considering a European call option maturing in  $T$  years and with strike  $K$ .  $g$  is the expected proportional change in the stock price caused by a jump.

$$g = \exp\left(\mu - \frac{1}{2}\gamma^2\right) - 1$$

$$\xi = \lambda(1 + g)$$

$$v = \ln(1 + g)$$

For  $k = 0, 1, 2 \dots$

$$\sigma_k^2 = \sigma^2 + \frac{1}{T} k \gamma^2$$

$$r_k = r - \lambda g + \frac{1}{T} k v$$

Merton shows that the price of the call option under the jump diffusion is:

$$c^{JD} = \sum_{k=0}^{\infty} \frac{e^{-\xi t} * (\xi t)^k}{k!} c^{BS}(S, K, T, r_k, \sigma_k)$$

The derivation of this formula is not quite as straightforward as the Black-Scholes formula. It is not possible to set up a portfolio that continuously replicates the option. Replication aims to use positions in the stock to track changes in the value of the option. If the stock price has unexpected jump moves, since the position in the stock responds linearly to changes in the stock price but the option responds nonlinearly, replication becomes impossible. Merton's approach is to assume that jump risk is diversifiable and is not priced. Under this assumption, Merton derives a mixed partial differential-difference equation that option prices must satisfy.

### 2.3. Implied volatility skew under jump diffusion model

The important motivation behind the development of the jump diffusion model is the presence of the implied volatility skew in options markets. Considering the aforementioned formulas we have three possibilities for skewness:

When  $\mu = 0$ , there is no skewness in the stock returns distribution; skewness is positive when  $\mu > 0$  and is negative when  $\mu < 0$ . From the option prices we back out the implied volatilities at various strike prices the results are presented in following figures. The range of strike prices used is symmetric around the current level of stock price.

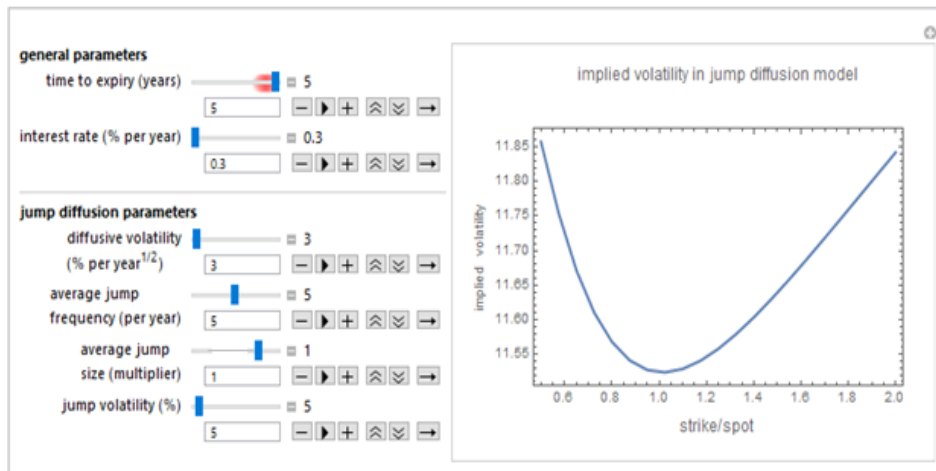


Figure 4: The implied volatility skew under jump diffusion model when  $\mu > 0$

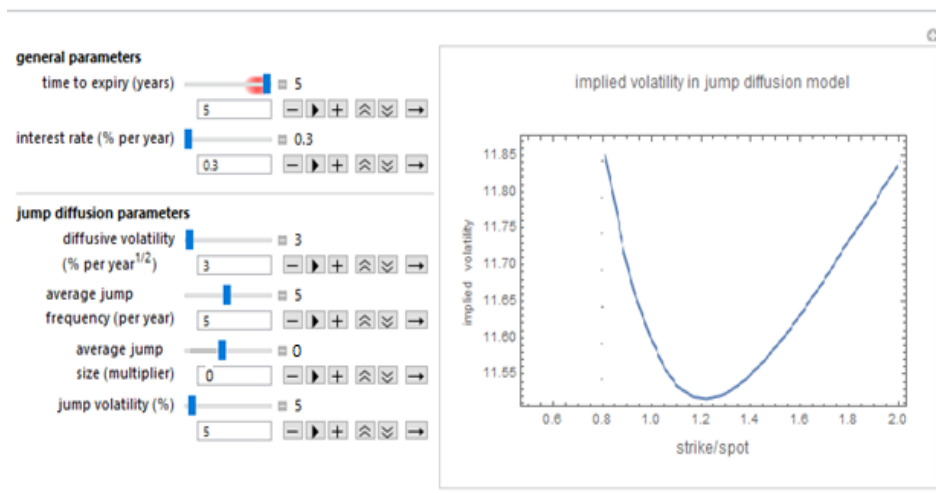


Figure 5: The implied volatility skew under jump diffusion model when  $\mu = 0$

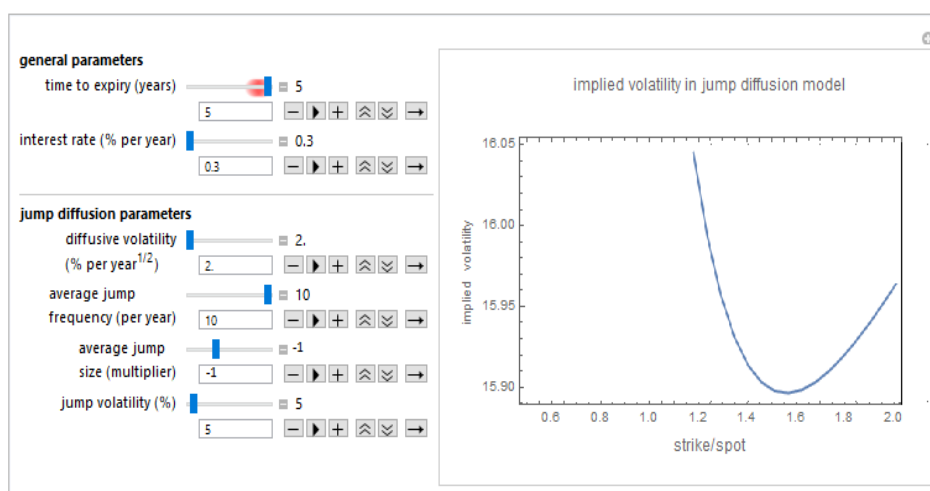


Figure 6: The implied volatility skew under jump diffusion model when  $\mu < 0$

The figures show that away-from-the-money options under jump diffusions generally have higher implied volatilities than at-the-money options, there is an implied volatility skew. When  $\mu = 0$ , there is no skewness and the implied volatility skew is symmetric. When  $\mu < 0$  the negative skewness in the returns distribution skews the implied volatility curve so that out-of-the-money options puts register higher implied volatilities than at-the-money options or out-of-the-money calls. For  $\mu > 0$ , the positive skewness means that the implied volatility curve is skewed to the right with higher implied volatilities for out-of-the-money calls than for at-the-money options or for out-of-the-money puts.

Jorion (1988) has found some support for the model in equity and currency markets. The ability of model to generate skewness and excess kurtosis means that it is typically able to match observed option prices, particularly at short maturities therefore returns become approximately normal as maturity increases. As a result the implied volatility smile under jump diffusion model becomes flat very rapidly, much faster than observed in practice.

### 3. Stochastic volatility model

Like jump-diffusion model the stochastic volatility model makes a single modification to the Black-Scholes Model. Volatility is allowed to evolve over time according to a separate stochastic process. This

time varying volatility creates fat tails in the returns distribution and address one of disadvantages of Black-Scholes Model. However, price paths are continuous in the stochastic model therefore it does not consider market gap.

For using this model we should specify three elements. 1. The underlying stock return process 2. The stochastic process of changing in volatility and 3. The correlation between changes in volatility and stock returns.

Many stochastic volatility models have been proposed in the literature. Most of them are continuous time models and few of them are discrete time models.

#### 3.1. Binomial-based stochastic volatility model

As we know the stock price in future will be calculated by following equations:

$$S_{t+h} = \begin{cases} S_{t+h}^u = e^{\sigma\sqrt{h}}S_t \\ S_{t+h}^d = e^{-\sigma\sqrt{h}}S_t \end{cases}$$

In the Black-Scholes Model we consider volatility as a constant but in stochastic volatility models we specify volatility for each period, by doing this we allow the

up and down moves to change from period to period, therefore our equations will be:

$$S_{t+h} = \begin{cases} S_{t+h}^u = e^{\sigma_t \sqrt{h}} S_t \\ S_{t+h}^d = e^{-\sigma_t \sqrt{h}} S_t \end{cases}$$

For calculating randomly-changing volatility we must first specify the stochastic process for the evolution of volatility over time and then use this to build the tree. The following formulas are a discrete time version of the model of Heston (1993):

$$V_{t+h}^u = V_t + \kappa(\theta - \sigma_t)h + \eta\sqrt{V_t h}$$

$$V_{t+h}^d = V_t + \kappa(\theta - \sigma_t)h - \eta\sqrt{V_t h}$$

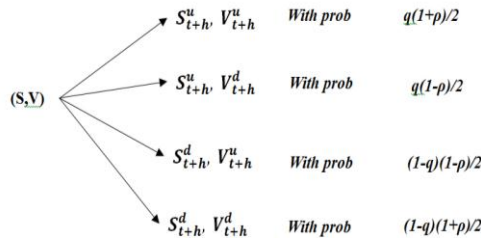
$\sigma_t$  denote the realized period t volatility, the term  $\kappa(\theta - \sigma_t)h$  called the drift of the process. The term  $\eta\sqrt{V_t h}$  represents the randomness in the evolution, the parameter  $\theta$  represents the mean long-run variance.

If current variance is less than  $\theta$ , the drift increases the variance while if current variance is greater than this level the drift decreases the variance. This specification for  $V_t$ , exhibits mean reversion. The term  $\kappa$  is the coefficient of mean reversion, it controls the speed which variance reverts to its mean. The higher is  $\kappa$  the faster is variance pushed towards its mean level  $\theta$ . the parameter  $\eta$  is called volatility of volatility. In a nutshell, the model has 3 crucial elements: a long term mean level around which volatility evolves, a coefficient of mean reversion and a volatility of volatility.

Since both the stock price and volatility can go up or down there is a total of four possible outcomes at time t+h:

$$(S_{t+h}^u, V_{t+h}^u), (S_{t+h}^u, V_{t+h}^d), (S_{t+h}^d, V_{t+h}^u), (S_{t+h}^d, V_{t+h}^d)$$

With binomial tree we have:



### 3.2. Continuous – time stochastic volatility models

Different Continuous – time formulations of stochastic volatility have been proposed. One the best known model is Heston’s model (1993). There are three equations that go into a stochastic volatility model description. One for the evolution of stock prices, one for the evolution of volatility and the final one describing the relation between first two.

The first equation in Heston’s model is:

$$dS_t = \alpha S_t dt + \sigma_t S_t dW_t^1$$

Here  $\alpha$  is the drift of the stock price process and  $W_t^1$  is the Brownian motion process.

The second equation in Heston’s model is:

$$dv_t = \kappa(\theta - v_t)dt + \eta\sqrt{v_t}dW_t^2$$

And the third one is:

$$E[dW_t^1 dW_t^2] = \rho dt$$

Kurtosis in stock returns in this model is created by random changes in volatility. Nonzero correlation between changes in volatility and returns results in skewness.

### 3.3. Option pricing under stochastic volatility

When volatility and returns are uncorrelated, Hull and White show that option prices in a stochastic volatility model may be expressed as a function of Black-Scholes process. We consider  $\bar{V}$  as an average variance over the life of the option.  $\bar{V}$  Will depend on the particular path of realized variances. Therefore  $h(\bar{V})$  denote the probability density function of  $\bar{V}$ . Hull and White specify following formula as a call price formula:

$$C^{sv} = \int_0^\infty C^{BS}(\bar{V})h(\bar{V})d\bar{V}$$

The general case where volatility and returns may be correlated is much harder and was solved in closed form in Heston’s paper (1993). Heston’s paper had a significant impact on option pricing because it opened



new technical approach to obtaining closed form solutions for option models, one that extended the basic setting of Black-Scholes formula and allowed for rapid computation of option prices in extended models. There are two ways we can derive option prices in continuous time setting. One is by using arbitrage arguments to reduce the option pricing problem to the solution to a partial difference equation. The other is by taking expectations under the risk neutral measure. The former approach, in most cases, defies closed form solutions. The latter involves solving for the expectation of the call payoff  $\max\{S_t - K, 0\}$  under the risk neutral probability and discounting this back to the present time. That is denoting by  $r$  the risk-free interest rate and by  $f$  the risk neutral stock price density at  $T$  conditional on current information, the call price is:

$$C = e^{-rT} \int_K^{\infty} (S_t - K) f(S_t) d(S_t)$$

The key innovation in Heston's paper was showing that this option price could be solved under stochastic volatility by solving to pdes, one each for the analogs of  $N(d_1)$  and  $N(d_2)$ . For more details, you can refer to python programming attached in this paper.

#### 4. Implied volatility based on the Heston price

The plots show the volatility surface generated by the Heston stochastic volatility model. This is implied volatility based on the Heston price, which depends on the time to expiration and on moneyness. Recall that for a call option, moneyness is the ratio of spot price to strike price. The Heston model is described by the following stochastic differential equations (SDE):

$$(dS)/S = \mu dt + dW_t,$$

$$dV_t = \kappa(V_{\infty} - V_t) dt + \sigma\sqrt{V_t}dZ_t$$

Where  $dW_t$  and  $dZ_t$  are correlated Brownian motions with,  $dW_t, dZ_t \rho$ .

The spot price follows the process with drift  $\mu$  and variance  $ZV_t$ , which is itself a stochastic process defined by the second equation. The second SDE is mean-reverting (the Cox–Ingersoll–Ross model, similar to the Ornstein–Hollenbeck process). Here the long-term variance is  $V_{\infty}$ , the mean reversion (or "speed of reversion") is  $\kappa$ , and the volatility of variance is  $\sigma$ . And finally there is another parameter that does not appear in the SDE, the initial condition for variance evolution (Heston 1993). For more illustration and visualization, we have simulated surface volatility and the following figures are samples which are selected from simulated population with corresponding parameters (Corresponding codes are appended):

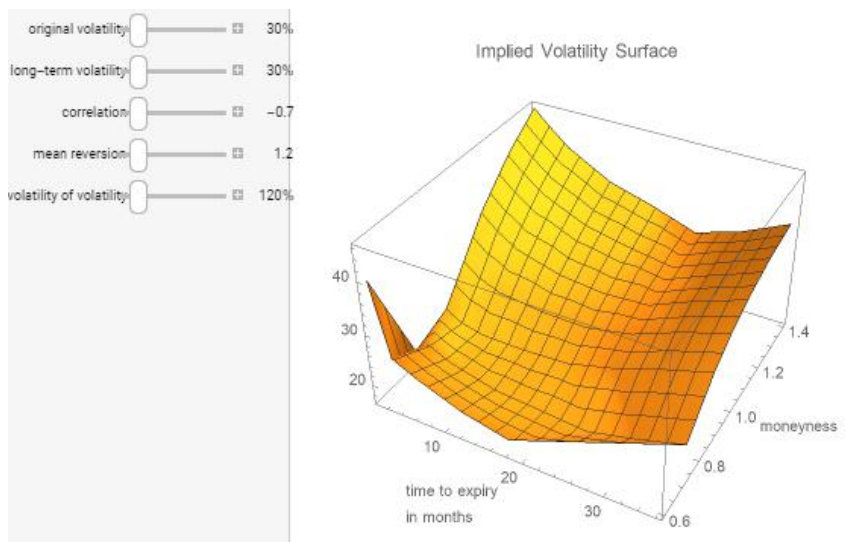


Figure 7: The implied volatility surface under stochastic volatility model with Low values for parameters

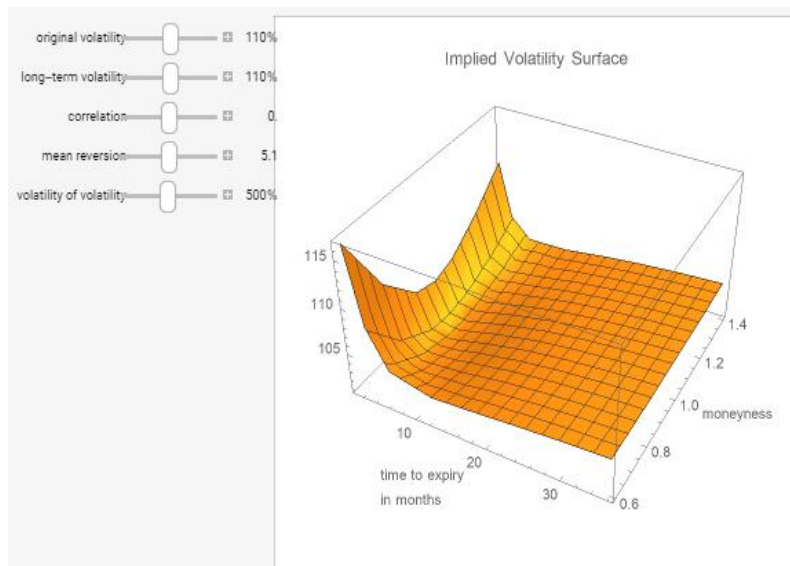


Figure 8: The implied volatility surface under stochastic volatility model with Middle values for parameters

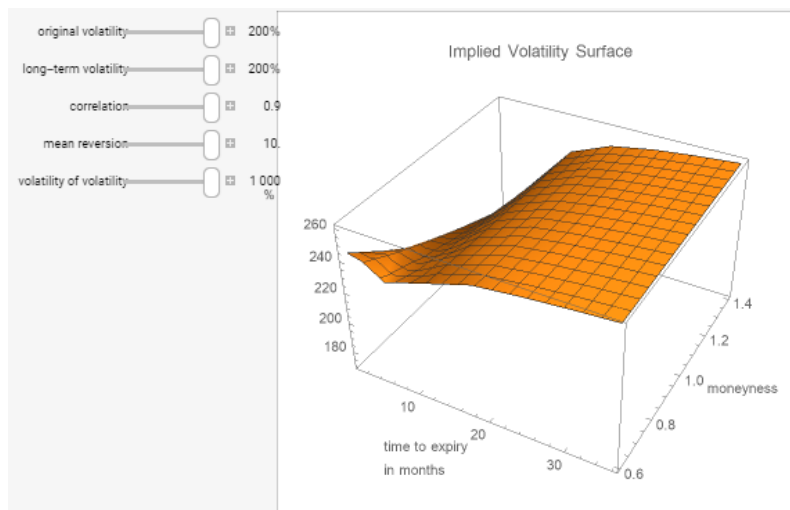


Figure 9: The implied volatility surface under stochastic volatility model with High values for parameters

As it is shown different parameters and their combinations with together can transform the volatility surface figure significantly, therefore, one should try to specify these parameters as accurate as possible.

### 5. Calibration of jump diffusion model

According to the earlier parts of this paper and provided background, Merton incorporated jump

process in his model. The following figure shows Merton Jump diffusion model which has coded in Python using corresponding parameters. It is clear that in this model there are gaps which are influenced by market fluctuations.

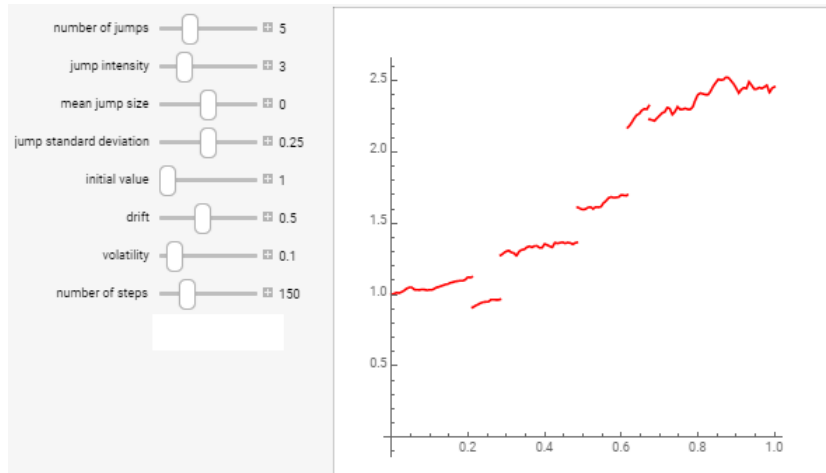


Figure 10: Calibration of jump diffusion model

Now, it is time to calculate option price using jump diffusion model. One approach to calibrating any model is to take the prices of traded options and to search over the model's parameter values so as to best match the prices of the options. This is the implied parameter approach. In the Black-Scholes Model the only unobserved parameter –the volatility- can be backed out of the price of a single option. However, in the jump diffusion model there are four unobserved parameters that need to be fit: the volatility of diffusion ( $\sigma$ ), the jump probability ( $\lambda$ ), the mean of the jump ( $\mu$ ) and the variance of the jump ( $\gamma^2$ ). For calculating call option price using jump diffusion model we have two approach: 1. We can use Monte Carlo simulation, or 2. Estimating each of parameters with different methods.

Using Monte Carlo Simulation we need three parameters: 1. Spot price, 2. Time to maturity and 3. Risk free rate. For example consider the following table.

With Python and MATLAB programming codes call price is 5.36 (it can be calculated by appended codes). As a second method we can calibrate each of parameters aforementioned and then put them in Merton jump diffusion model. We have calculated option price with this method and get 6.13 (it can be calculated by appended codes) comparing to real price of 6.25 and Black-Scholes estimate of 5.9. The difference between Black-Scholes model and jump diffusion model in option pricing has shown in the following figure.

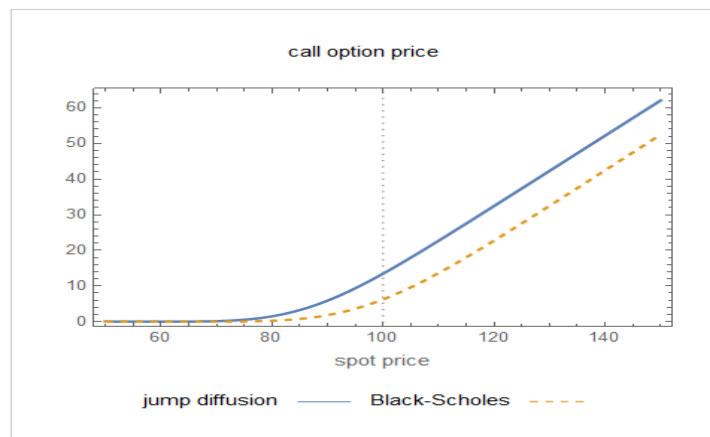


Figure 11: Call option price calculated by jump diffusion Vs. Black-Scholes model

**Table 2: Example of jump diffusion model**

Parameters	Values	Monte Carlo Simulation	Estimation
S0	100	100	100
X	101	-	101
R	0.1	0.1	0.1
Sig	0.25	-	0.25
Mu	0	-	0
Gam	0.5	-	0.5
Lambda	0.1	-	0.1
T	0.25	0.25	0.25

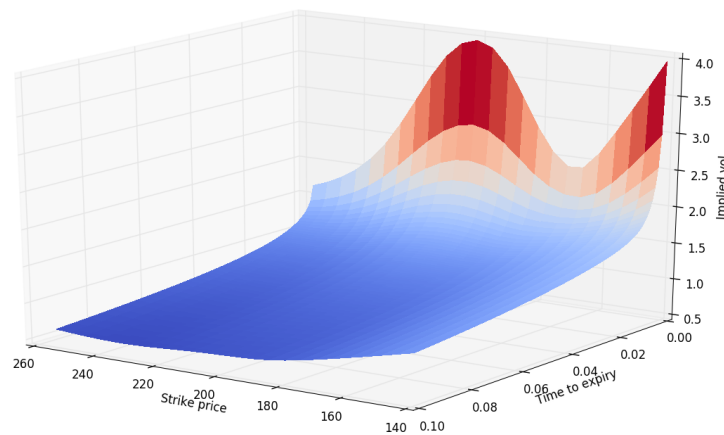
### 6. Calibration of the stochastic volatility model

There are five unknown variables in the Heston model:  $\sigma$ ,  $\kappa$ ,  $\theta$ ,  $\eta$  and  $\rho$ . One method for estimating these values is to find the parameters that minimize the sum of square difference between the fitted implied

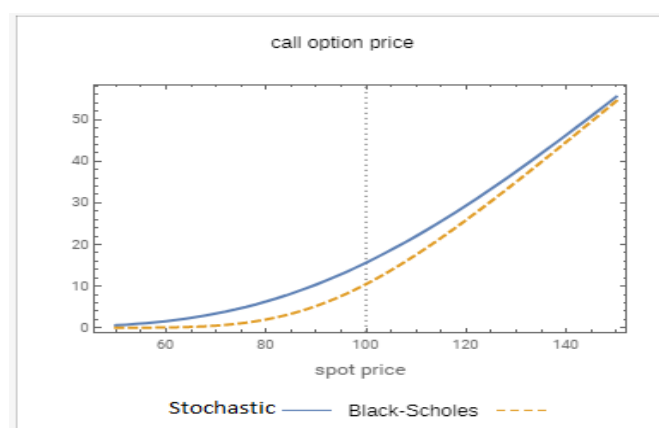
volatility and those observed in market prices. Other method is to minimize the sum of absolute deviations of model and market implied volatility. In order to visualize Heston model in three dimensions we used following parameters to run our Python code and demonstrate it.

**Table 3: Example of stochastic volatility model**

S	200
v	0.8
r	0.1
dividend	0
Kappa	1
Theta	0.4
Eta	15
Rho	-0.5
Up	0.1
Down	-0.1
UpSigma	0.02
DownSigma	0.02
StepSize	0.4



**Figure 8: Visualization of Heston model**



**Figure 13: Call option price calculated by stochastic Vs. Black-Scholes model**

At low maturities changing volatility has not had enough time to create excess kurtosis, while at long maturities mean reversion eliminates excess kurtosis. Stochastic volatility models have only limited impact on short dated option prices. And the evidence that the option smile in equities markets remains steep even at very long maturities (Foresi and Wu, 2005) suggests that these models may not do well at matching the data unless other factors are also included in the models.

Empirically, as with jump diffusion model the evidence about stochastic volatility model is mixed. Varying degrees of support of such models have been found in different markets (Bates. 1996). The model's ability to generate skewness and excess kurtosis enables it to better fit observed option prices than the Black-Scholes model. However, stochastic volatility models imply a hump shaped pattern of excess kurtosis (Das and Sundaram, 1999).

## 7. Conclusion

This paper made effort to illustrate Jump diffusion and stochastic volatility models and concepts.

To achieve profound knowledge considering those models we modeled them in Python and MATLAB. As it has shown jump diffusion model could explain implied volatility which cannot be explained by Black-Scholes model and because it considers jumps between prices its results provide better estimation for realized option price. On the other hand, Stochastic volatility model not only considers implied volatility but also suppose the volatility is mutable, using simulation we concluded that this model also has better estimation power than Black-Scholes model to predict option

price. Each model obtains some improvement over Black-Scholes but none is also a completely satisfactory resolution of the non-normality problem. This has to led to the proposal of several further and technically more sophisticated alternatives such as the variance - gamma model and the log stable model as well as models based on stable-Paretian and inverse-Gaussian process.

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